# ON THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS OF SUCCESSIVE APPROXIMATIONS.

By PRESTON A. LAMBERT.

(*Read April 20, 1911.*)

The object of this paper is to apply to the solution of linear differential equations, both ordinary and partial, the method of expansion into series used in the solution of algebraic equations in the papers read by the author before the Philosophical Society in April, 1903, and in April, 1908.

Let the given differential equation be

$$(1) \qquad f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

The method of solution consists of the following steps:

(*a*) Break up the left-hand member of the differential equation into two parts,

$$f_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right)$$

and

$$f_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right),$$

such that the first part equated to zero can be integrated by some known method, and multiply the second part by a parameter $S$, independent of $x$ and $y$. Replace the given equation by

$$(2) \quad f_1\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^2}\right) + Sf_2\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0.$$

(*b*) Assume that

$$(3) \qquad y = y_0 + y_1S + y_2S^2 + y_3S^3 + y_4S^4 + \cdots$$

makes equation (2) an identity.

($c$) In this identity arranged according to the ascending powers of $S$ equate to zero the coefficients of the different powers of $S$.

($d$) Solve the differential equations thus obtained in regular order for $y_0, y_1, y_2, y_3, y_4, \cdots$.

($e$) Substitute these values in (3) and make $S$ unity. The resulting value of $y$, if it contains a finite number of terms or if it is a uniformly convergent infinite series, is a solution of the given differential equation.[1]

The method of solution of linear differential equations as here outlined does not seem to occur in mathematical literature except as developed by the author.

The method will be exemplified by applying it to two differential equations, important in mathematical physics—Bessel's equation, a second order ordinary differential equation, and Fourier's equation for the flow of heat, a second order partial differential equation.

Bessel's equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0.$$

Replace Bessel's equation by

$$\left(x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y\right) + Sx^2y = 0$$

and assume that

$$y = y_0 + y_1S + y_2S^2 + y_3S^3 + y_4S^4 + \cdots$$

makes the latter equation an identity.

When arranged in ascending powers of $S$ this identity is

$$\begin{array}{llll} x^2\dfrac{d^2y_0}{dx^2} & + x^2\dfrac{d^2y_1}{dx^2} & S + x^2\dfrac{d^2y_2}{dx^2} & S^2 + \cdots \equiv 0. \\ + x\dfrac{dy_0}{dx} & + x\dfrac{dy_1}{dx} & + x\dfrac{dy_2}{dx} & \\ - n^2y_0 & - n^2y_1 & - n^2y_2 & \\ & + x^2y_0 & + x^2y_1 & \end{array}$$

[1] This method gives a formal solution of non-linear differential equations, but up to the present time the author has been unable to test the resulting series for convergency.

Equating to zero the coefficients of the powers of $S$ in this identity, there result the following differential equations for the determination of $y_0, y_1, y_2, y_3, \cdots$.

$$x^2 \frac{d^2y_0}{dx^2} + x\frac{dy_0}{dx} - n^2y_0 = 0,$$

$$x^2 \frac{d^2y_1}{dx^2} + x\frac{dy_1}{dx} - n^2y_1 + x^2y_0 = 0,$$

$$x^2 \frac{d^2y_2}{dx^2} + x\frac{dy_2}{dx} - n^2y_2 + x^2y_1 = 0.$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

The equation in $y_0$ is a homogeneous linear differential equation and its solution is

$$y_0 = Ax^n + Bx^{-n}.$$

Substituting this value of $y_0$ the equation for determining $y_1$ becomes

$$x^2 \frac{d^2y_1}{dx^2} + x\frac{dy_1}{dx} - n^2y_1 = - Ax^{n+2} - Bx^{-n+2}.$$

This equation becomes exact when multiplied by $x^{-n-1}$. The resulting equation integrated gives a linear equation of the first order, the solution of which is

$$y_1 = \frac{-Ax^{n+2}}{2^2(n+1)} + \frac{Bx^{-n+2}}{2^2(n-1)}.$$

Substituting this value of $y_1$ in the equation for determining $y_2$ and proceeding in the same manner

$$y_2 = \frac{Ax^{n+4}}{2^4 \cdot 2\,!\,(n+1)(n+2)} + \frac{Bx^{-n+4}}{2^4 \cdot 2\,!\,(n-1)(n-2)}.$$

In like manner

$$y_3 = \frac{-Ax^{n+6}}{2^6 \cdot 3\,!\,(n+1)(n+2)(n+3)} + \frac{Bx^{-n+6}}{2^6 \cdot 3\,!\,(n-1)(n-2)(n-3)},$$

and so on.

Substituting these values of $y_0, y_1, y_2, y_3, \cdots$ in

$$y = y_0 + y_1 S + y_2 S^2 + y_3 S^3 + y_4 S^4 + \cdots$$

and making $S$ unity,

$$y = Ax^n \left[ 1 - \frac{1}{n+1}\frac{x^2}{2^2} + \frac{1}{(n+1)(n+2)}\frac{x^4}{2^4 \cdot 2!} - \frac{1}{(n+1)(n+2)(n+3)}\frac{x^6}{2^6 \cdot 3!} + \cdots \right]$$

$$+ Bx^{-n}\left[ 1 + \frac{1}{n-1}\frac{x^2}{2^2} + \frac{1}{(n-1)(n-2)}\frac{x^4}{2^4 \cdot 2!} + \frac{1}{(n-1)(n-2)(n-3)}\frac{x^6}{2^6 \cdot 3!} + \cdots \right].$$

When $n$ is not an integer the terms of both series in this value of $y$ continue indefinitely according to the law of formation which inspection makes evident, both series are uniformly convergent except when $x = 0$, and both series are solutions of the given differential equation.

When $n$ is a negative integer the law of formation of the terms of the first series changes after the $(n)$th term and when $n$ is a positive integer the law of formation of the terms of the second series changes after the $(n)$th term. The second case will be considered.

When $n$ is a positive integer the $(n)$th term of the second series is

$$y_{n-1} = \frac{Bx^{n-2}}{2^{2(n-1)}(n-1)!\,(n-1)!}.$$

Substituting this value of $y_{n-1}$ in the differential equation for determining $y_n$,

$$x^2 \frac{d^2 y_n}{dx^2} + x\frac{dy_n}{dx} - n^2 y_n + x^2 y_{n-1} = 0,$$

and solving for $y_n$ by the method used in solving for $y_1, y_2, y_3, \cdots$,

$$y_n = \frac{B}{2^{2n-1} n!\,(n-1)!}\left[ x^n \log x - \frac{x^n}{2n} \right].$$

In determining $y_{n+1}, y_{n+2}, y_{n+3}, \cdots$, the second term in the bracket gives the terms of the first series in the value of $y$ multiplied by a constant. This new series is combined with the first series in the value of $y$.

The first term in the bracket gives

$$y_{n+1} = \frac{-B}{2^{2n-1}n!(n-1)!}\left[-\frac{x^{n+2}\log x}{2^2(n+1)} + \frac{x^{n+2}}{2^2(n+1)}\left(1 + \frac{1}{n+1}\right)\right],$$

$$y_{n+2} = \frac{-B}{2^{2n-1}n!(n-1)!}\left[\frac{x^{n+4}\log x}{2!\,2^4(n+1)(n+2)} - \frac{x^{n+4}}{2!\,2^4(n+1)(n+2)}\left(1 + \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}\right)\right].$$

. . . . . . . . . . .

The solution of Bessel's differential equation when $n$ is a positive integer is therefore

$$\begin{aligned} y = Ax^n\Bigg[1 &- \frac{1}{n+1}\frac{x^2}{2^2} + \frac{1}{(n+1)(n+2)}\frac{x^4}{2^4\cdot 2!} - \frac{1}{(n+1)(n+2)(n+3)}\frac{x^6}{2^6\cdot 3!} + \cdots\Bigg] \\ &+ Bx^{-n}\left[1 + \frac{1}{n-1}\frac{x^2}{2^2} + \frac{1}{(n-1)(n-2)}\frac{x^4}{2^4\cdot 2!} + \cdots + \frac{x^{2n-2}}{2^{2(n-1)}(n-1)!(n-1)!}\right] \\ &- \frac{Bx^n\log x}{2^{2n-1}n!(n-1)!}\left[1 - \frac{1}{n+1}\frac{x^2}{2^2} + \frac{1}{(n+1)(n+2)}\frac{x^4}{2^4\cdot 2!} - \frac{1}{(n+1)(n+2)(n+3)}\frac{x^6}{2^6\cdot 3!} + \cdots\right] \\ &- \frac{Bx^n}{2^{2n-1}n!(n-1)!}\left[\frac{1}{n+1}\left(1 + \frac{1}{n+1}\right)\frac{x^2}{2^2} - \frac{1}{(n+1)(n+2)}\left(1 + \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2}\right)\frac{x^4}{2^4\cdot 2!} + \cdots\right]. \end{aligned}$$

This is also the solution of the differential equation when $n$ is a negative integer.

Fourier's partial differential equation for the linear flow of heat is

$$\frac{\partial V}{\partial t} = K \frac{\partial^2 V}{\partial x^2}.$$

Replace Fourier's equation by

$$\frac{\partial V}{\partial t} = SK \frac{\partial^2 V}{\partial x^2}$$

and assume that

$$V = V_0 + V_1 S + V_2 S^2 + V_3 S^3 + \cdots$$

makes the latter equation an identity.

When arranged in ascending powers of $S$ this identity is

$$\frac{\partial V_0}{\partial t} + \left| \begin{array}{l} \dfrac{\partial V_1}{\partial t} \\ - K \dfrac{\partial^2 V_0}{\partial x^2} \end{array} \right| S + \left| \begin{array}{l} \dfrac{\partial V_2}{\partial t} \\ - K \dfrac{\partial^2 V_1}{\partial x^2} \end{array} \right| S^2 + \left| \begin{array}{l} \dfrac{\partial V_3}{\partial t} \\ - K \dfrac{\partial^2 V_2}{\partial x^2} \end{array} \right| S^3 + \cdots \equiv 0.$$

Equating to zero the coefficient of the powers of $S$ in this identity, there result the following partial differential equations for the determination of $V_0, V_1, V_2, V_3, \cdots$,

$$\frac{\partial V_0}{\partial t} = 0, \qquad \frac{\partial V_1}{\partial t} - K \frac{\partial^2 V_0}{\partial x^2} = 0,$$

$$\frac{\partial V_2}{\partial t} - K \frac{\partial^2 V_1}{\partial x^2} = 0, \qquad \frac{\partial V_3}{\partial t} - K \frac{\partial^2 V_2}{\partial x^2} = 0,$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

These partial differential equations solved in regular order give

$$V_0 = \phi(x), \quad V_1 = \phi''(x)(Kt), \quad V_2 = \phi^{\text{IV}}(x)\frac{(Kt)^2}{2\,!},$$

$$V_3 = \phi^{\text{VI}}(x)\frac{(Kt)^3}{3\,!}, \cdots.$$

Substituting these values of $V_0, V_1, V_2, V_3, \ldots$ in the assumed value of $V$ and finally making $S$ unity, there results

$$(A) \quad V = \phi(x) + \phi''(x)(Kt) + \phi^{\text{IV}}(x)\frac{(Kt)^2}{2\,!} + \phi^{\text{VI}}(x)\frac{(Kt)^3}{3\,!} + \cdots,$$

which is a solution of Fourier's equation for all values of $\phi(x)$ for which $V$ either contains a finite number of terms or is an infinite series uniformly convergent both in $x$ and in $t$.

The following table shows several values of $\phi(x)$ and the corresponding solutions of Fourier's equation,

| | | |
|---|---|---|
| I | (1) $\phi(x) = A,$ | $V = A,$ |
| | (2) $\phi(x) = Ax,$ | $V = Ax,$ |
| | (3) $\phi(x) = Ax^2,$ | $V = A(x^2 + 2Kt),$ |
| | (4) $\phi(x) = A \sin(nx),$ | $V = Ae^{-n^2Kt} \sin(nx),$ |
| | (5) $\phi(x) = A \cos(nx),$ | $V = Ae^{-n^2Kt} \cos(nx),$ |
| | (6) $\phi(x) = Ae^{nx},$ | $V = Ae^{nx+n^2Kt},$ |
| | (7) $\phi(x) = Ae^{-nx},$ | $V = Ae^{-nx+n^2Kt},$ |
| | (8) $\phi(x) = Ae^{nx} \sin(nx),$ | $V = Ae^{nx} \sin(nx + 2n^2Kt).$ |

It will be noticed that in these solutions $\phi(x)$ is the value of $V$ when $t = 0$, that is $V = \phi(x)$ is the initial heat distribution.

It will also be noticed that in all these results $x$ may be replaced by $x + a$. This statement is true of the results in the several following tables.

If Fourier's differential equation is replaced by

$$S\frac{\partial V}{\partial t} = K\frac{\partial^2 V}{\partial x^2}$$

and the assumption made that

$$V = V_0 + V_1 S + V_2 S^2 + V_3 S^3 + \cdots$$

makes this equation an identity, this identity arranged in ascending powers of $S$ is

$$K\frac{\partial^2 V_0}{\partial x^2}\Bigg| + \left. \begin{matrix} K\dfrac{\partial^2 V_1}{\partial x^2} \\ -\dfrac{\partial V_0}{\partial t} \end{matrix} \right| S + \left. \begin{matrix} K\dfrac{\partial^2 V_2}{\partial x^2} \\ -\dfrac{\partial V_1}{\partial t} \end{matrix} \right| S^2 + \left. \begin{matrix} K\dfrac{\partial^2 V_3}{\partial x^2} \\ -\dfrac{\partial V_2}{\partial t} \end{matrix} \right| S^3 + \cdots \equiv 0.$$

Equating to zero the coefficients of the powers of $S$ in this identity,

$$\frac{\partial^2 V_0}{\partial x^2} = 0, \quad \frac{\partial^2 V_1}{\partial x^2} - \frac{1}{K}\frac{\partial V_0}{\partial t} = 0, \quad \frac{\partial^2 V_2}{\partial x^2} - \frac{1}{K}\frac{\partial V_1}{\partial t} = 0, \quad \cdots.$$

Solving these partial differential equations in regular order for $V_0, V_1, V_2, V_3, \cdots$, substituting these values in the assumed expression for $V$, and finally making $S$ unity, the result

$$(B) \quad \begin{aligned} V = \phi(t)x + \frac{1}{K}\phi'(t)\frac{x^3}{3!} + \frac{1}{K^2}\phi''(t)\frac{x^5}{5!} + \cdots \\ + \theta(t) + \frac{1}{K}\theta'(t)\frac{x^2}{2!} + \frac{1}{K^2}\theta''(t)\frac{x^4}{4!} + \cdots \end{aligned}$$

is a solution of Fourier's differential equation for all values of $\phi(t)$ and $\theta(t)$ for which $V$ either contains a finite number of terms or is an infinite series uniformly convergent both for $x$ and for $t$.

Solutions of the differential equation when $\phi(t) = 0$ corresponding to several values of $\theta(t)$ are as follows—

II $\quad \phi(t) = 0$,

(1) $\theta(t) = A, \qquad V = A,$

(2) $\theta(t) = At, \qquad V = A\left(t + \frac{x^2}{2K}\right),$

(3) $\theta(t) = At^2, \qquad V = A\left(t^2 + \frac{x^2 t}{K} + \frac{x^4}{3\cdot 4K^2}\right),$

(4) $\theta(t) = At^{\frac{1}{2}}, \qquad V = At^{\frac{1}{2}}\left[1 + \frac{x^2}{2!(2Kt)} - \frac{x^4}{4!(2Kt)^2} + \frac{3x^6}{6!(2Kt)^3} - \cdots\right],$

(5) $\theta(t) = At^{-\frac{1}{2}}, \qquad V = \frac{Ae^{-\frac{x^2}{4Kt}}}{Kt^{\frac{1}{2}}},$

(6) $\theta(t) = At^{\frac{3}{2}}, \qquad V = At^{\frac{3}{2}}\left[1 + \frac{3}{2!}\frac{x^2}{2Kt} + \frac{3}{4!}\frac{x^4}{(2Kt)^2} - \frac{3}{6!}\frac{x^6}{(2Kt)^3} - \cdots\right],$

(7) $\theta(t) = At^{-\frac{3}{2}}, \qquad V = At^{-\frac{3}{2}}\left[1 - \frac{3}{2!}\frac{x^2}{2Kt} + \frac{3\cdot 5}{4!}\frac{x^4}{(2Kt)^2} - \frac{3\cdot 5\cdot 7}{6!}\frac{x^6}{(2Kt)^3} + \cdots\right],$

(8) $\theta(t) = A\sin(nt)$, $V = A\left[\sin(nt) + \frac{n}{K}\cos nt\,\frac{x^2}{2!} - \frac{n^2}{K^2}\sin(nt)\,\frac{x^4}{4!} - \cdots\right]$,

(9) $\theta(t) = A\log t$, $V = A\left[\log t + \frac{1}{Kt}\,\frac{x^2}{2!} - \frac{1}{K^2t^2}\,\frac{x^4}{4!} + \frac{2}{K^3t^3}\,\frac{x^6}{6!} - \frac{2\cdot 3}{K^4t^4}\,\frac{x^8}{8!} + \cdots\right]$,

(10) $\theta(t) = Ae^{nt}$, $V = Ae^{nt}\left[1 + \frac{n}{K}\,\frac{x^2}{2!} + \frac{n^2}{K^2}\,\frac{x^4}{4!} + \cdots\right]$.

It will be noticed that in these solutions $V = \theta(t)$ is the heat distribution when $x = 0$.

Solutions of the differential equation when $\theta(t) = 0$ corresponding to several values of $\phi(t)$ are as follows:

III $\theta(t) = 0$,

(1) $\phi(t) = A$, $V = Ax$,

(2) $\phi(t) = At$, $V = A\left[xt + \frac{x^3}{3!\,K}\right]$,

(3) $\phi(t) = At^2$ $V = A\left[xt^2 + \frac{2x^3t}{3!\,K} + \frac{2x^5}{5!\,K^2}\right]$,

(4) $\phi(t) = At^{\frac{1}{2}}$ $V = At^{\frac{1}{2}}\left[x + \frac{1}{2Kt}\,\frac{x^3}{3!} - \frac{1}{2^2K^2t^2}\,\frac{x^5}{5!} + \frac{3}{2^3K^3t^3}\,\frac{x^7}{7!} - \cdots\right]$,

(5) $\phi(t) = At^{-\frac{1}{2}}$, $V = At^{-\frac{1}{2}}\left[x - \frac{1}{2Kt}\,\frac{x^3}{3!} + \frac{3}{2^2K^2t^2}\,\frac{x^5}{5!} - \frac{3\cdot 5}{2^3K^3t^3}\,\frac{x^7}{7!} + \cdots\right]$,

(6) $\phi(t) = Ae^{nt}$, $V = Ae^{nt}\left[x + \frac{n}{K}\,\frac{x^3}{3!} + \frac{n^2}{K^2}\,\frac{x^5}{5!} + \cdots\right]$,

(7) $\phi(t) = A\sin(nt)$, $V = A\left[\sin(nt)x + \frac{n}{K}\cos(nt)\frac{x^3}{3!} - \frac{n^2}{K^2}\sin(nt)\frac{x^5}{5!} - \cdots\right]$,

(8) $\phi(t) = A \log t, \quad V = A\left[x \log t + \frac{1}{Kt}\frac{x^3}{3!} - \frac{1}{K^2t^2}\frac{x^5}{5!} + \frac{2}{K^3t^3}\frac{x^7}{7!} - \cdots\right].$

It will be noticed that in this set of solutions $V = 0$ when $x = 0$.

Let $u_1 = f_1(x, t)$, $u_2 = f_2(y, t)$, $u_3 = f_3(z, t)$ represent solutions of the three one-dimensional Fourier's equations,

$$\frac{\partial V}{\partial t} = K\frac{\partial^2 V}{\partial x^2}, \quad \frac{\partial V}{\partial t} = K\frac{\partial^2 V}{\partial y^2}, \quad \frac{\partial V}{\partial t} = K\frac{\partial^2 V}{\partial z^2}$$

respectively. It is readily proved that

$$V = u_1 u_2 \text{ and } V = u_1 u_2 u_3$$

are solutions respectively of the two-dimensional Fourier's equation

$$\frac{\partial V}{\partial t} = K\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right)$$

and the three-dimensional Fourier's equation

$$\frac{\partial V}{\partial t} = K\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right).$$

This shows how solutions of the two- and three-dimensional Fourier's equations can be obtained from the solutions of the one-dimensional equation.

For example, from the one-dimensional solutions

$$V = \frac{Ae^{-\frac{x^2}{4Kt}}}{Kt^{\frac{1}{2}}} \quad \text{and} \quad V = Ae^{-n^2Kt}\sin(nx)$$

the three-dimensional solutions

IV (1) $\quad V = \dfrac{Ae^{-\frac{r^2}{4Kt}}}{K^3t^{\frac{3}{2}}},$

(2) $\quad V = Ae^{-(\alpha^2+\beta^2+\gamma^2)Kt}\sin(\alpha x)\sin(\beta y)\sin(\gamma z),$

respectively, are obtained.

If the solution of the three-dimensional Fourier's equation

$$\frac{\partial V}{\partial t} = K\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right)$$

is a function of $r$ and $t$ only, so that

$$V = f(r, t), \text{ where } r = (x^2 + y^2 + z^2)^{\frac{1}{2}},$$

the transformation of the given equation from rectangular to polar coördinates shows that the solution is

$$V = \frac{u}{r},$$

where $u$ is a solution of the Fourier's equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial r^2}.$$

It follows that solutions of the three-dimensional equation of the form $V = f(r, t)$ are obtained by replacing $x$ by $r$ in any solution of the one-dimensional equation

$$\frac{\partial V}{\partial t} = K\frac{\partial^2 u}{\partial x^2},$$

and dividing the result by $r$.

In this manner are obtained the solutions

V (1) $$V = \frac{A}{r},$$

(2) $$V = \frac{Ae^{\frac{-r^2}{4Kt}}}{rKt^{\frac{1}{2}}},$$

(3) $$V = \frac{A}{r}e^{nr}\sin(nr + 2n^2Kt).$$

It is interesting to compare the solutions of Fourier's partial differential equation obtained in this paper with the solutions tabulated by Sir William Thomson in the mathematcal appendix of the article on "Heat" in the "Encyclopaedia Britannica," ninth edition.

Sir William Thomson obtains his results by summation, that is by integration, from the solution IV (1) above. All his results occur directly in the above tables or are combinations of two of these solutions. It is evident that there are several misprints in the results as printed in the "Britannica."

Of course there are many solutions of Fourier's equation which must be built up from elementary solutions, however found, by means of Fourier series, or which must be obtained by the methods of harmonic analysis.

The solution III (5) above is the series used by Sir William Thomson in his solution of the problem of the secular cooling of the earth.[2]

An interesting result in pure mathematics is obtained as follows: Sir William Thomson shows that for a continued point source of heat, if the rate is an arbitrary function of the time, $f(t)$, the solution of Fourier's equation when $K = 1$ is given by the definite integral

$$V = \int_0^\infty dx f(t - x) \frac{e^{\frac{-r^2}{4x}}}{8\pi^{\frac{3}{2}} x^{\frac{3}{2}}}.$$

The second part of the general solution (B) above shows that

$$V = \frac{1}{4\pi}\left[\frac{1}{r} f(t) + f'(t)\frac{r}{2!} + f''(t)\frac{r^3}{4!} + \cdots\right]$$

is also the solution of Fourier's equation for the same conditions.

It follows that

$$\int_0^\infty dx f(t - x) \frac{e^{\frac{-r^2}{4x}}}{8\pi^{\frac{3}{2}} x^{\frac{3}{2}}} = \frac{1}{4\pi}\left[\frac{1}{r} f(t) + f'(t)\frac{r}{2!} + f''(t)\frac{r^3}{4!} + \cdots\right]$$

is a general formula for computing the definite integral.

Lehigh University,
Bethlehem, Pa.

[2] "Mathematical and Physical Papers," Vol. III.